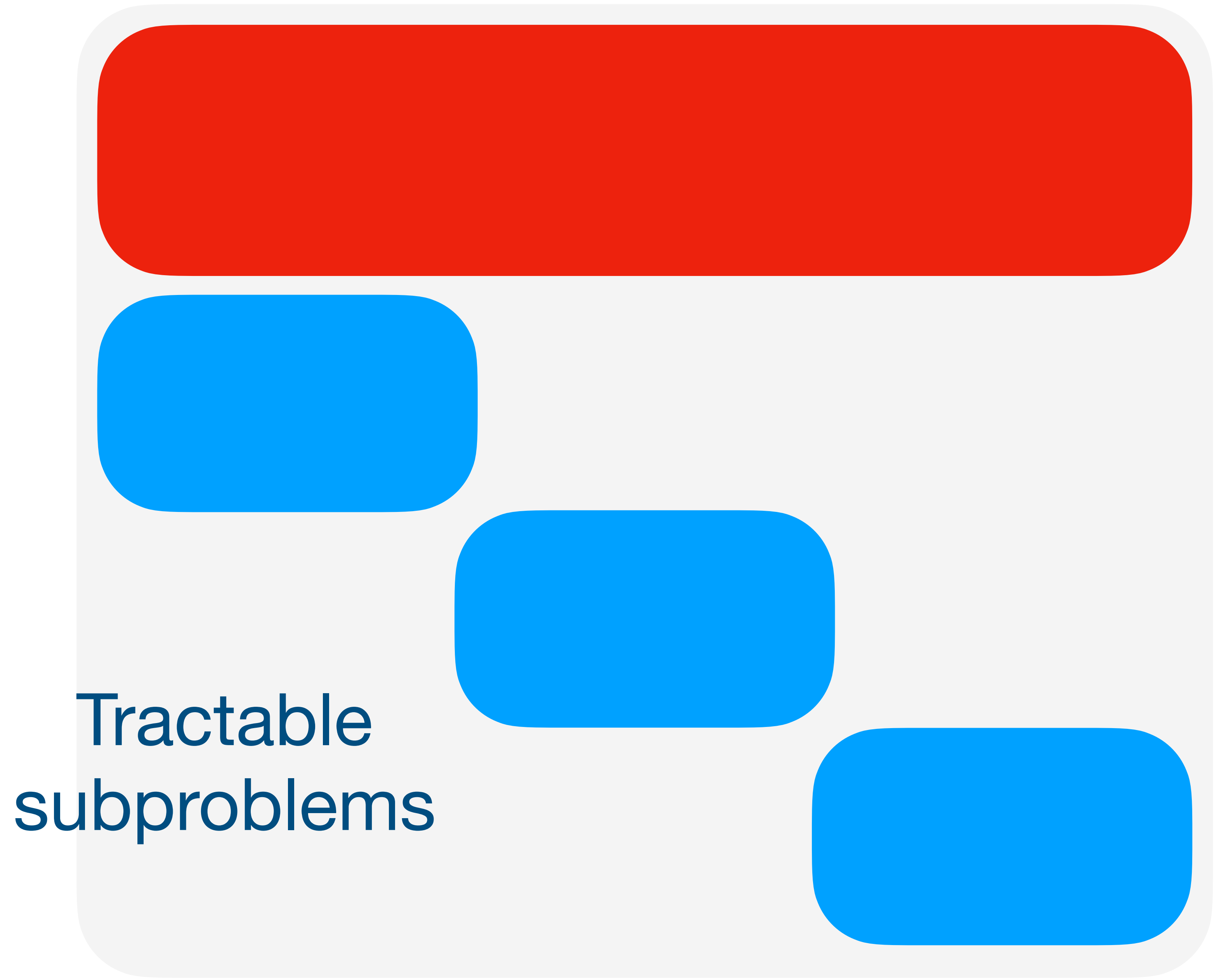
Building Coluna.jl, a branch-cut-and-price framework in Julia

Natacha Javerzat, Guillaume Marques, Vitor Nesello, Artur Pessoa, Ruslan Sadykov, and François Vanderbeck

Coluna.jl

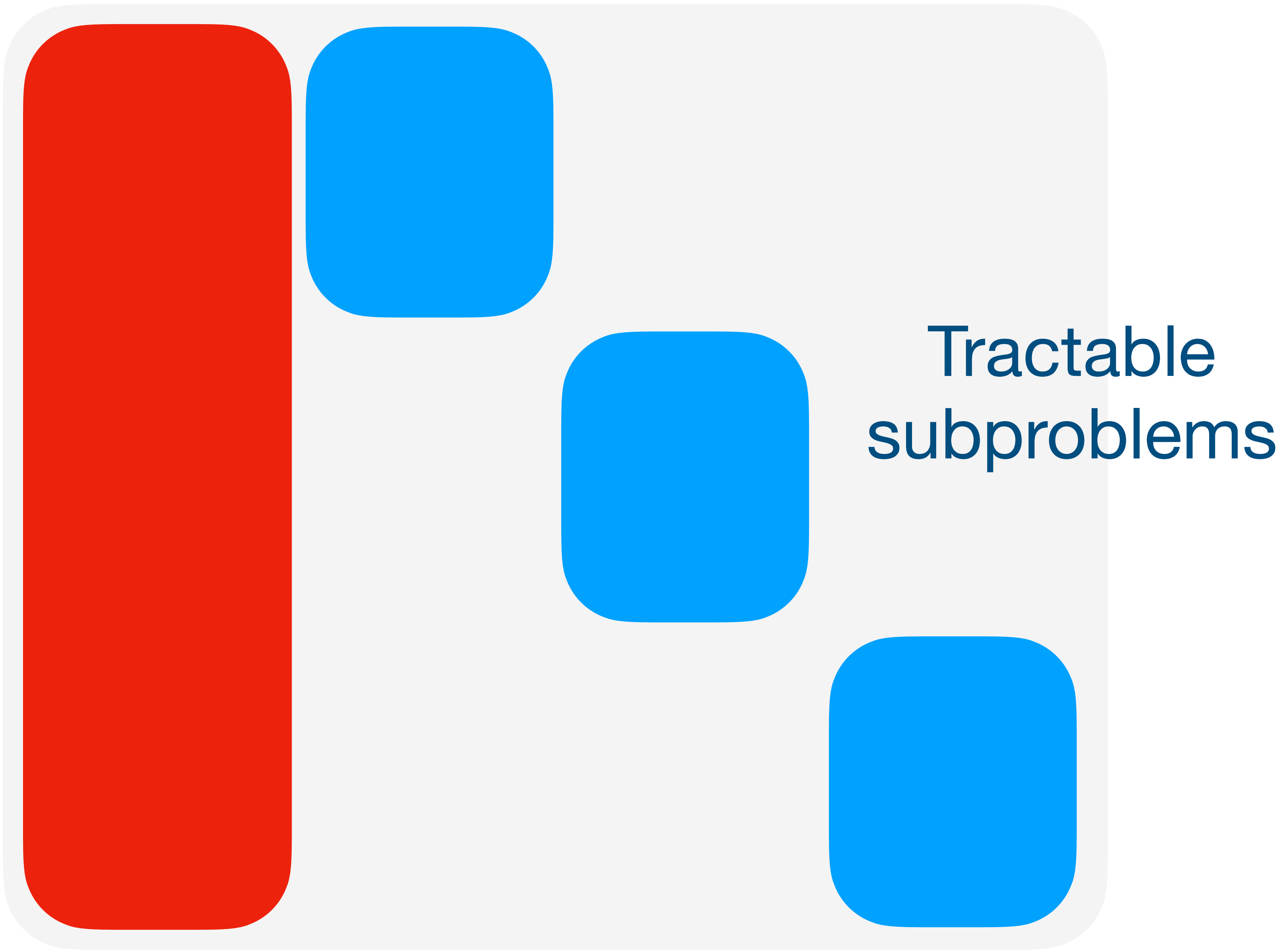
A Julia framework to optimize block-structured MILP problems.

Linking constraints



Dantzig-Wolfe decomposition

Linking variables

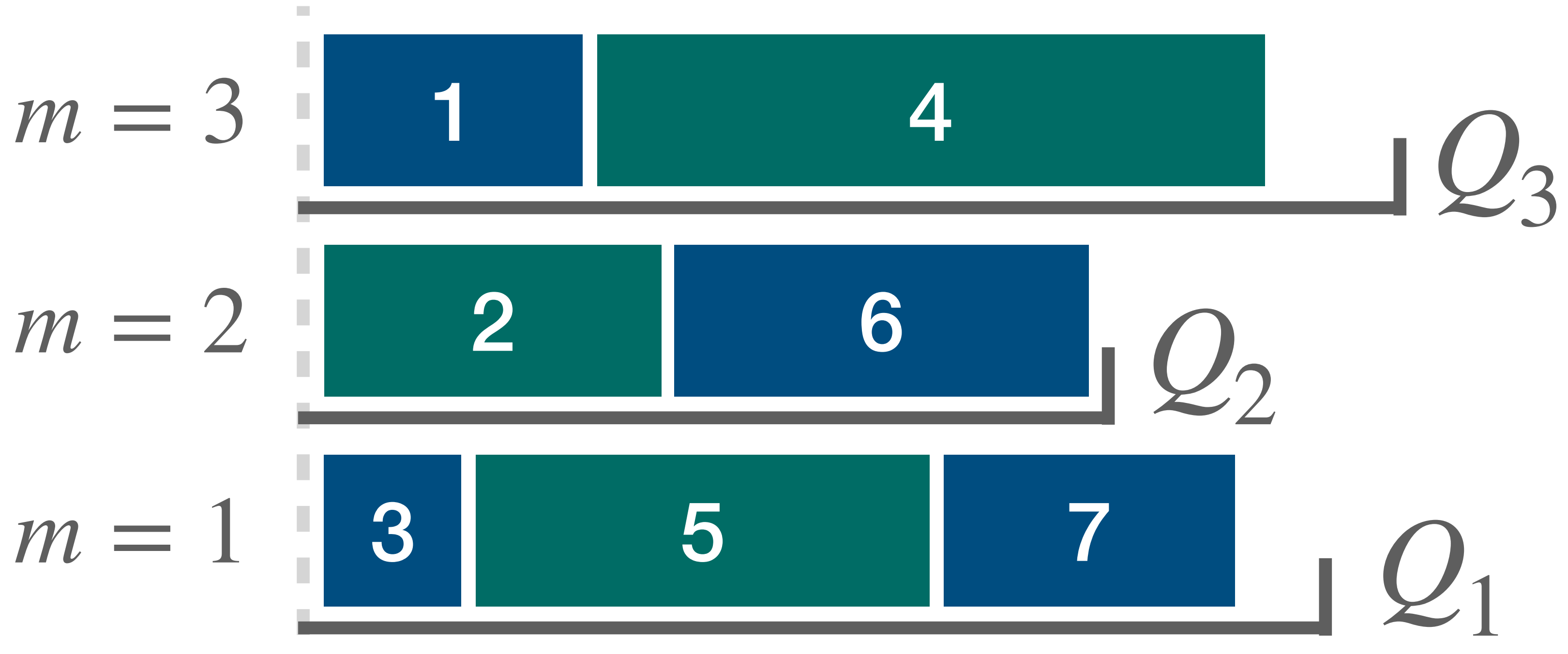


Benders decomposition

Generalized Assignment Problem

Let:

$x_{mj} = 1$	if job j assigned to machine m; 0 otherwise
c_{mj}	cost
w_{mj}	weight
Q_m	capacity of machine m



Example of solution

$$\min \sum_{m \in M} \sum_{j \in J} c_{mj} x_{mj}$$

s.t. $$\sum_{m \in M} x_{mj} \geq 1 \qquad j \in J$$

$$\sum_{i \in I} w_{mj} x_{mj} \leq Q_m \qquad m \in M$$

$$x_{mj} \in \{0,1\}$$

Generalized Assignment Problem

model.jl

```julia
using JuMP, GLPK

M = 1:3 # Machines
J = 1:15 # Jobs

model = Model(GLPK.Optimizer)

@variable(model, x[m in M, j in J], Bin)

@constraint(model, cov[j in J], sum(x[m, j] for m in M) >= 1)
@constraint(model, knp[m in M],
  sum(w[m, j] * x[m, j] for j in J) <= Q[m]
)

@objective(model, Min, sum(c[m, j] * x[m, j] for m in M, j in J));
```

Generalized Assignment Problem

model.jl

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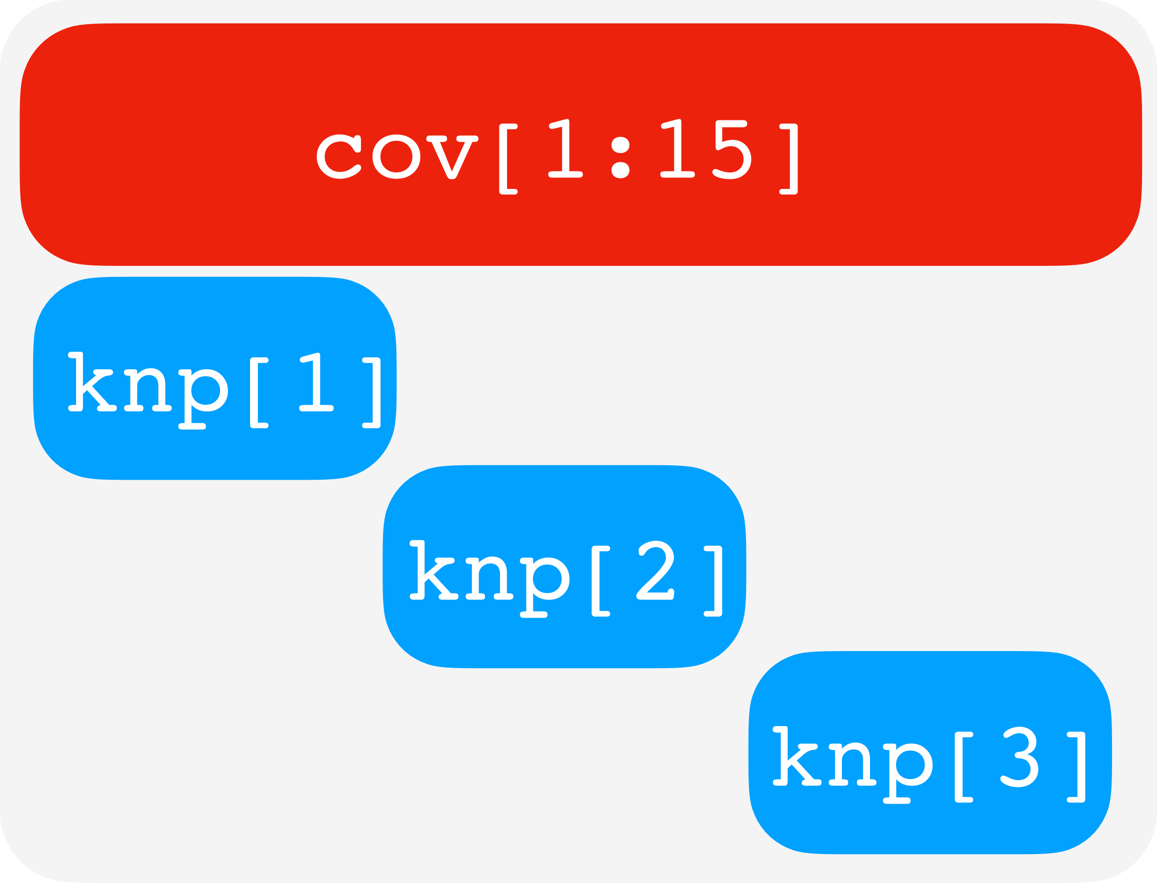
```



Original formulation

Generalized Assignment Problem

model.jl

```
using JuMP, GLPK, BlockDecomposition, Coluna

@axis(M, 1:3) # Annotated machines
J = 1:15 # Jobs
coluna = optimizer_with_attributes(Coluna.Optimizer, ...)

model = BlockModel(coluna)

@variable(model, x[m in M, j in J], Bin)

@constraint(model, cov[j in J], sum(x[m, j] for m in M) >= 1)
@constraint(model, knp[m in M],
  sum(w[m, j] * x[m, j] for j in J) <= Q[m]
)

@objective(model, Min, sum(c[m, j] * x[m, j] for m in M, j in J));
@dantzig_wolfe_decomposition(model, decomposition, M)
```

Generalized Assignment Problem

```julia
using JuMP, HiGHS

M = 1:3 # Machines
J = 1:15 # Jobs

model = Model(HiGHS.Optimizer)

@variable(model, x[m in M, j in J], Bin)
@constraint(model, cov[j in J], sum(x[m, j] for m in M) >= 1)
@constraint(model, knp[m in M],
    sum(w[m,j] * x[m, j] for j in J) <= Q[m]
)
@objective(model, Min, sum(c[m, j] * x[m, j] for m in M, j in J));
```

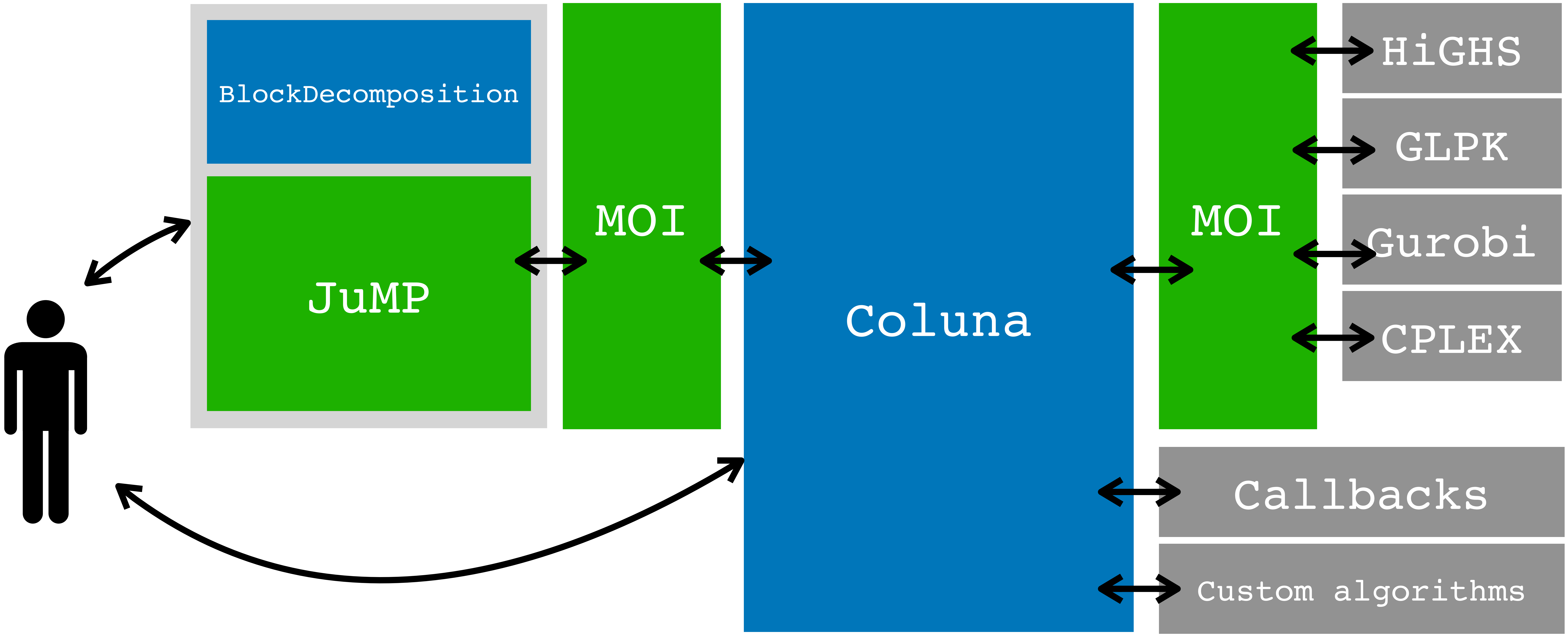
```julia
using JuMP, HiGHS, BlockDecomposition, Coluna

@axis(M, 1:3) # Annotated machines
J = 1:15 # Jobs
coluna = optimizer_with_attributes(Coluna.Optimizer, ...)

model = BlockModel(coluna)

@variable(model, x[m in M, j in J], Bin)
@constraint(model, cov[j in J], sum(x[m, j] for m in M) >= 1)
@constraint(model, knp[m in M],
    sum(w[m,j] * x[m, j] for j in J) <= Q[m]
)
@objective(model, Min, sum(c[m, j] * x[m, j] for m in M, j in J));
@dantzig_wolfe_decomposition(model, decomposition, M)
```

Coluna environment



BlockDecomposition.jl

src/JuMP.jl

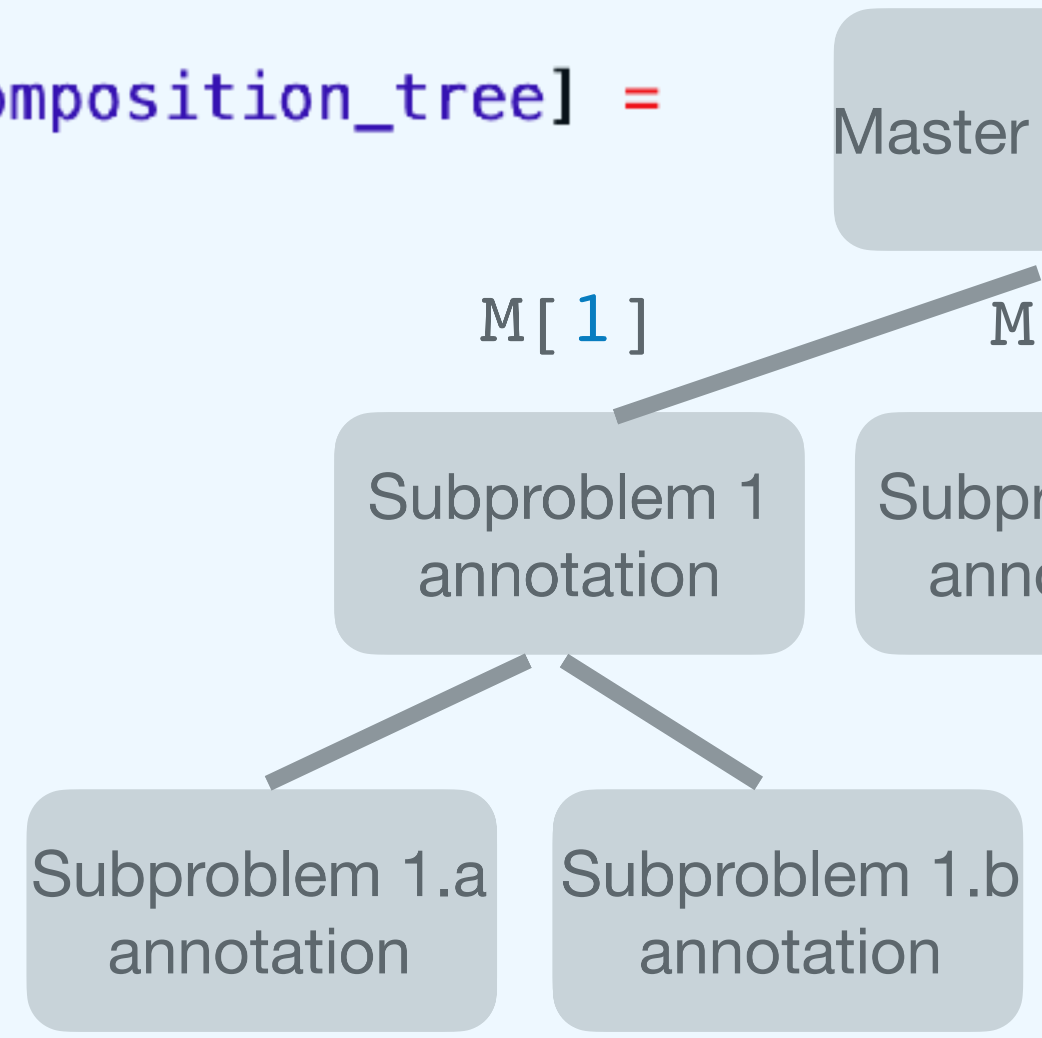
```julia
mutable struct GenericModel{T<:Real} <: AbstractModel
    # [...]
    optimize_hook::Any
    # [...]
    ext::Dict{Symbol,Any}
    # [...]
end
```

When calling `BlockModel`:

```julia
function optimize_hook!(m::JuMP.Model)
    # [...] automatic decomposition if activated
    register_decomposition(m)
    return JuMP.optimize!(m, ignore_optimize_hook = true)
end
```

When calling `@dantzig_wolfe_decomposition`:

```julia
model.ext[:decomposition_tree] =
```



Backlog: **nested decomposition**

Danztig-Wolfe reformulation

When calling `JuMP.optimize!`

- BlockDecomposition hook annotates all the variables and constraints
- Coluna receives the formulation and the annotations from JuMP/MathOptInterface
- Coluna reformulates the formulation

```
mutable struct Reformulation
    parent::Formulation{Original}
    master::Formulation{DwMaster}
    dw_pricing_subprs::Dict{FormId, Formulation{DwSp}}
    # [...]
end
```

`cov[1:15]` $\sum_{m \in M} \sum_{k \in K^m} \tilde{x}^k_{mj} \lambda^m_k \geq 1 \quad j \in J$

$\sum_k \lambda^m_k \leq 1 \quad m \in M$

$\lambda^m_k \geq 0$

master formulation

λ^m_k is nb of times solution $\tilde{x}^k \in K^m$ to the subproblem `knp[m]` is used.

`knp[1]` `knp[2]` `knp[3]`

pricing subproblems

- Coluna calls the « top algorithm »

Benders reformulation

- Use an `@axis` to define the index-set of subproblems
- Write the compact model with JuMP
- Call the `@benders_decomposition` macro

$$\begin{aligned} \min \quad & cy + fx \\ \text{s.t.} \quad & Ay \geq a \\ & Ty + Dx \geq b \\ & Ex \geq e \\ & x, y \geq 0 \end{aligned}$$

original formulation

$$\begin{aligned} \min \quad & cy + \sum_k \eta^k \\ \text{s.t.} \quad & Ay \geq a \\ & \langle \text{Benders cuts} \rangle \\ & \eta \in \mathbb{R}, y \geq 0 \end{aligned}$$

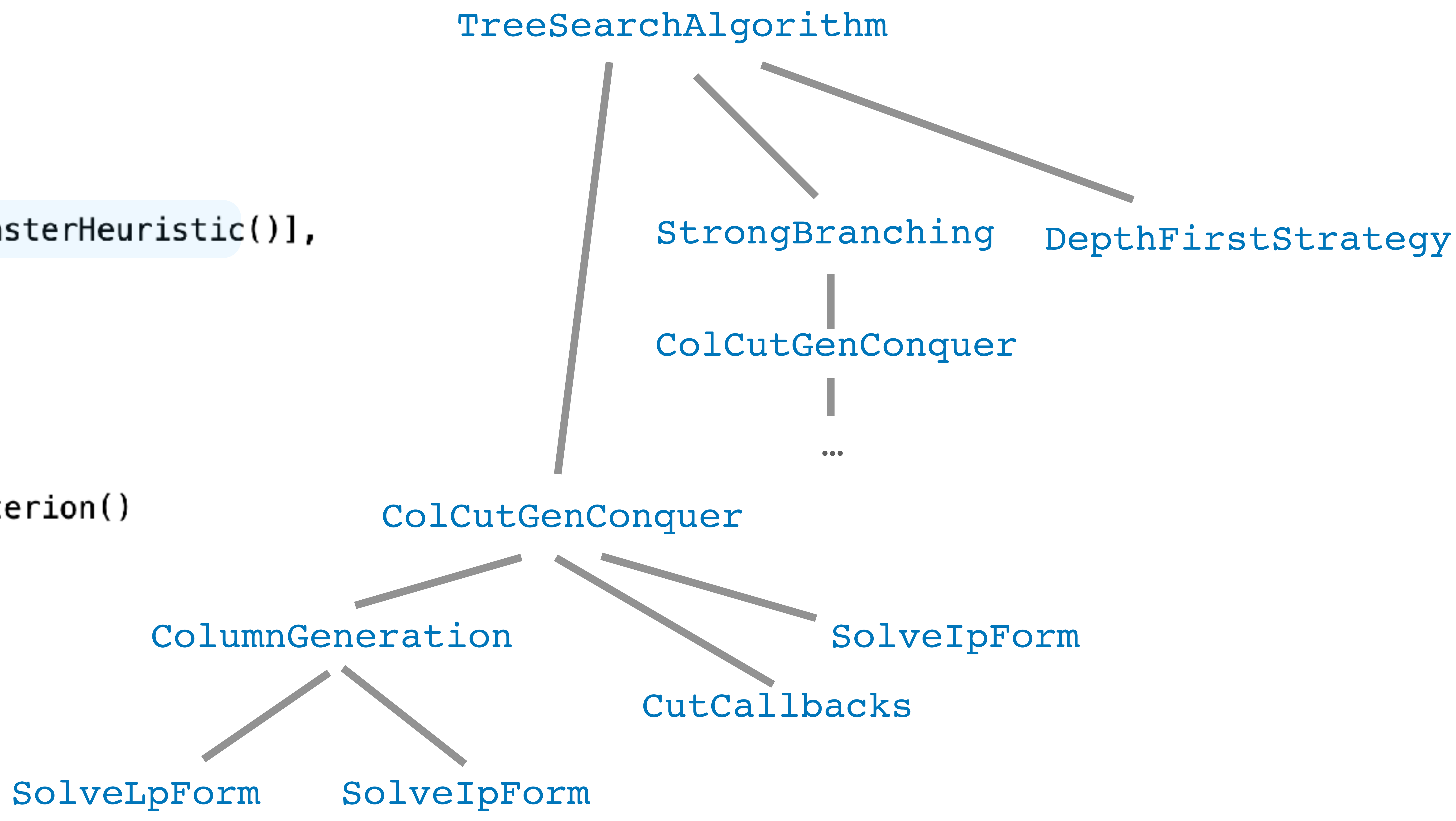
master formulation (1st level)

$$\begin{aligned} \min \quad & fx \\ \text{s.t.} \quad & Dx \geq b - T\bar{y} \\ & Ex \geq e \\ & x \geq 0 \end{aligned}$$

separation subproblems (2nd level)
where $\bar{x}$ is the first-level solution

Top algorithm of Coluna

```
bcp = Coluna.Algorithm.TreeSearchAlgorithm(
    conqueralg = ColCutGenConquer(
        colgen = ColumnGeneration(
            max_nb_iterations = 1000
        ),
        primal_heuristics = [DefaultRestrictedMasterHeuristic()],
    ),
    cutgen = CutCallbacks()
    dividealg = StrongBranching(
        phases = [...]
        rules = [...]
        selection_criterion = MostFractionalCriterion()
    ),
    explorestrategy = DepthFirstStrategy(),
    maxnumnodes::Int = 50,
    branchingtreefile = "tree.dot"
)
```



Branch-cut-and-price output

```
Coluna
Version 0.7.0 | https://github.com/atoptima/Coluna.jl
*****************************************************************************************
**** B&B tree root node
**** Local DB = -Inf, global bounds: [ -Inf , Inf ], time = 0.00 sec.
*****************************************************************************************
  <st= 1> <it=  1> <et= 0.00> <mst= 0.00> <sp= 0.00> <cols= 5> <al= 0.00> <DB=-129136.2200> <mlp=100000.0000> <PB=Inf>
  <st= 1> <it=  2> <et= 0.01> <mst= 0.00> <sp= 0.00> <cols= 5> <al= 0.00> <DB=-148537.5000> <mlp=30340.5500> <PB=Inf>
  <st= 1> <it=  3> <et= 0.01> <mst= 0.00> <sp= 0.00> <cols= 5> <al= 0.00> <DB=-139032.5600> <mlp=20319.9200> <PB=Inf>
  <st= 1> <it=  4> <et= 0.01> <mst= 0.00> <sp= 0.00> <cols= 5> <al= 0.00> <DB=-124563.8557> <mlp= 3253.8629> <PB=Inf>
  <st= 1> <it=  5> <et= 0.01> <mst= 0.00> <sp= 0.00> <cols= 5> <al= 0.00> <DB=  313.9200> <mlp=  410.1222> <PB=Inf>
  […]
  <st= 1> <it= 13> <et= 0.05> <mst= 0.00> <sp= 0.00> <cols= 1> <al= 0.00> <DB=  395.4817> <mlp=  396.4117> <PB=Inf>
  <st= 1> <it= 14> <et= 0.05> <mst= 0.00> <sp= 0.00> <cols= 0> <al= 0.00> <DB=  396.2954> <mlp=  396.2954> <PB=Inf>
Cut separation callback adds 0 new essential cuts and 1 new facultative cuts.
avg. viol. = 0.54, max. viol. = 0.54, zero viol. = 0.
  <st= 1> <it=  1> <et= 1.71> <mst= 0.00> <sp= 0.00> <cols= 2> <al= 0.00> <DB=  395.4634> <mlp=  399.3076> <PB=Inf>
  <st= 1> <it=  2> <et= 1.71> <mst= 0.00> <sp= 0.00> <cols= 1> <al= 0.00> <DB=  398.1515> <mlp=  399.0815> <PB=Inf>
  <st= 1> <it=  3> <et= 1.71> <mst= 0.00> <sp= 0.00> <cols= 0> <al= 0.00> <DB=  398.9050> <mlp=  398.9050> <PB=Inf>
Cut separation callback adds 0 new essential cuts and 1 new facultative cuts.
avg. viol. = 0.50, max. viol. = 0.50, zero viol. = 0.
  […]
  <st= 1> <it=  1> <et= 1.77> <mst= 0.00> <sp= 0.00> <cols= 5> <al= 0.00> <DB=-14668.8378> <mlp= 1528.5356> <PB=Inf>
  <st= 1> <it=  2> <et= 1.77> <mst= 0.00> <sp= 0.00> <cols= 4> <al= 0.00> <DB=  433.7789> <mlp=  444.2689> <PB=Inf>
  <st= 1> <it=  3> <et= 1.78> <mst= 0.00> <sp= 0.00> <cols= 3> <al= 0.00> <DB=  434.5150> <mlp=  440.3950> <PB=Inf>
  <st= 1> <it=  4> <et= 1.78> <mst= 0.00> <sp= 0.00> <cols= 3> <al= 0.00> <DB=  436.0250> <mlp=  440.3950> <PB=Inf>
  <st= 1> <it=  5> <et= 1.78> <mst= 0.00> <sp= 0.00> <cols= 4> <al= 0.00> <DB=  433.6600> <mlp=  440.3950> <PB=Inf>
  <st= 1> <it=  6> <et= 1.78> <mst= 0.00> <sp= 0.00> <cols= 2> <al= 0.00> <DB=  438.6150> <mlp=  440.3950> <PB=Inf>
  <st= 1> <it=  7> <et= 1.78> <mst= 0.00> <sp= 0.00> <cols= 0> <al= 0.00> <DB=  440.3950> <mlp=  440.3950> <PB=Inf>
Cut separation callback adds 0 new essential cuts and 0 new facultative cuts.
*****************************************************************************************
**** B&B tree node N°3, parent N°1, depth 1, 1 untreated node
**** Local DB = 440.3950, global bounds: [ 440.3950 , 444.4000 ], time = 1.78 sec.
**** Branching constraint: x[3,1]<=0.0
*****************************************************************************************
  <st= 1> <it=  1> <et= 1.78> <mst= 0.00> <sp= 0.00> <cols= 4> <al= 0.00> <DB=  433.7450> <mlp=  443.6750> <PB=444.4000>
  <st= 1> <it=  2> <et= 1.79> <mst= 0.00> <sp= 0.00> <cols= 3> <al= 0.00> <DB=  438.6300> <mlp=  441.2575> <PB=444.4000>
```

Algorithms provided by Coluna

Column-and-cut generation

- Column Generation (API)
  - Auto smoothing stabilization
  - Identical subproblems
  - Multi-stage
- Pricing callback
- Initial columns callback
- Lazy-cut callback
- User-cut callback
  - Robust cuts
  - Non-robust cuts
- Restricted master heuristic
- *Columns cleanup*

Based on [Pessoa et al., 2018], [Poggi de Aragão and Uchoa 2003], [Jepsen et al., 2006]

Benders cut generation (API)

- Multi-cut
- *Integration with B&B*
- *Stabilization*

Based on [Bonami et al. 2020]

Algorithms provided by Coluna

Strong Branching

- Selection criterion (API)
  - Most fractional
  - First found
  - Least Fractional
  - Closest To Non Zero Integer
- Scores (API)
  - Product score
  - Tree Depth score
- Rules (API)
  - Single variable

Inspired from [Pecin et al., 2017], [Le Bodic & George Nemhauser, 2017], [Achterberg 2007], [Kullmann, 2009}

Tree search (API)

- Depth-First Search
- Best-Bound Search

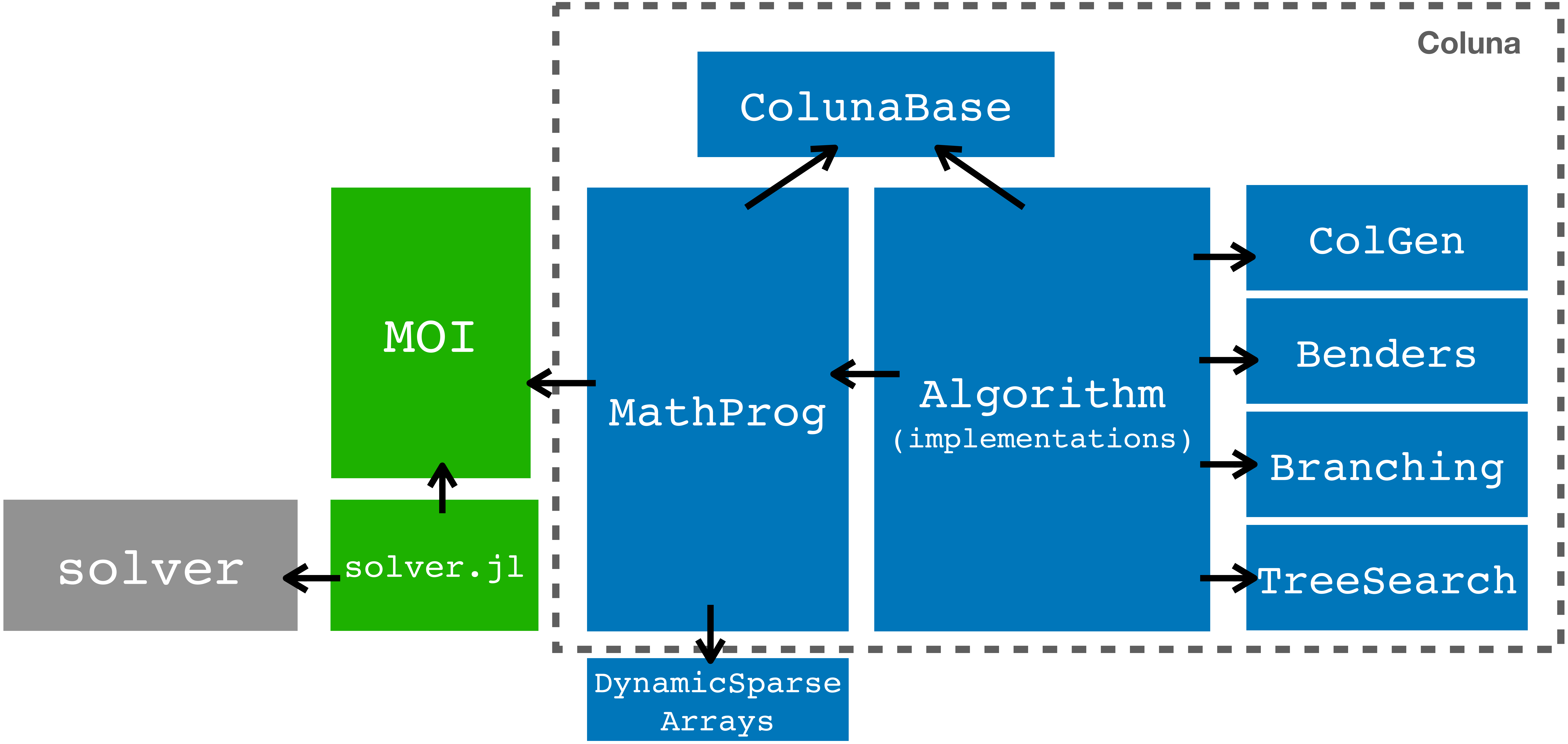
- *LDS (Draft PR)*

Presolve (dev)

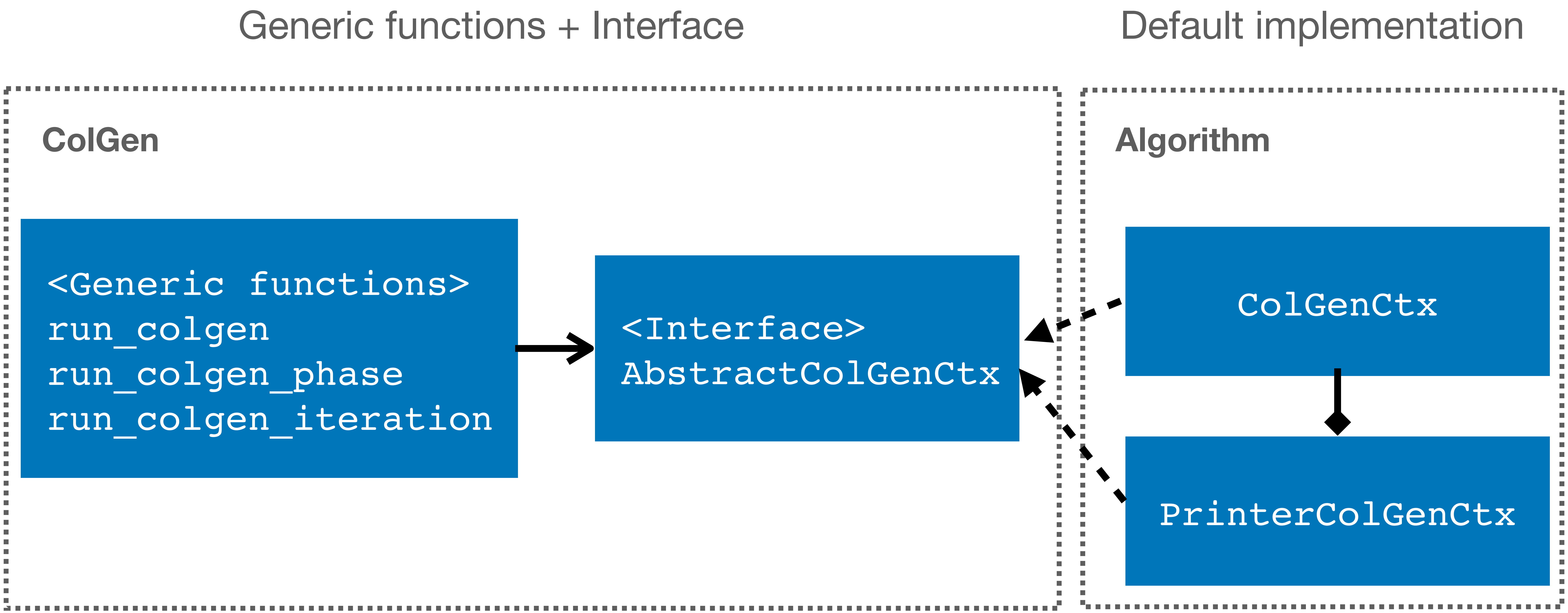
- Single row elimination
- Variable fixing
- Variable bounds strengthening
- Propagation between master/pricing subproblems

Inspired from [Achterberg et al., 2020] & [Unpublished work]

Coluna architecture



Algorithms architecture



Algorithms architecture

Generic functions + Interface

« textbook algorithm »

algorithmic logic

documented interface

```
@mustimplement "ColGen" update_reduced_costs!(
  context, phase, red_costs
) = nothing
```

Default implementation

« algorithm details »

algorithm runs
with the formulation representation
provided by `MathProg`

Algorithms architecture

Generic functions + Interface

« textbook algorithm »

algorithmic logic

documented interface

Default implementation

« algorithm details »

algorithm runs
with the formulation representation
provided by `MathProg`

Modular & easy to test

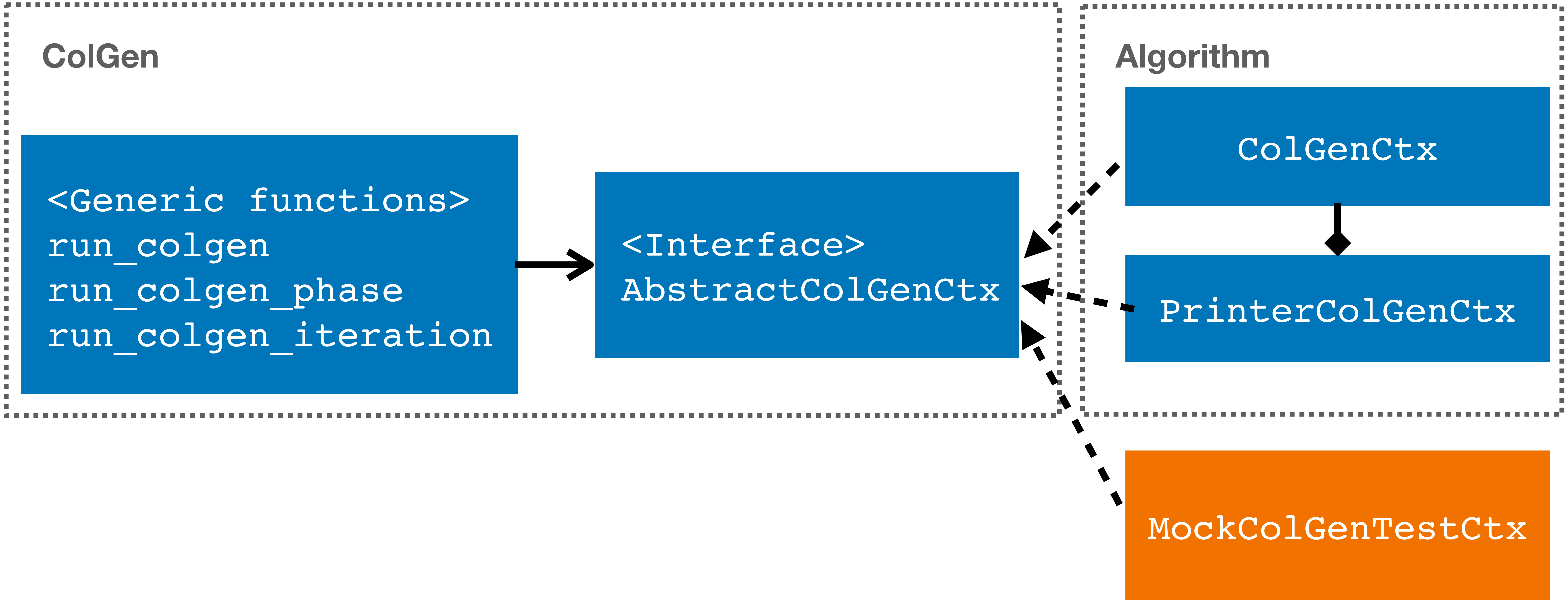
- unit tests of the logic with mocks
- e2e tests

- unit tests
- integration tests with `MathProg`

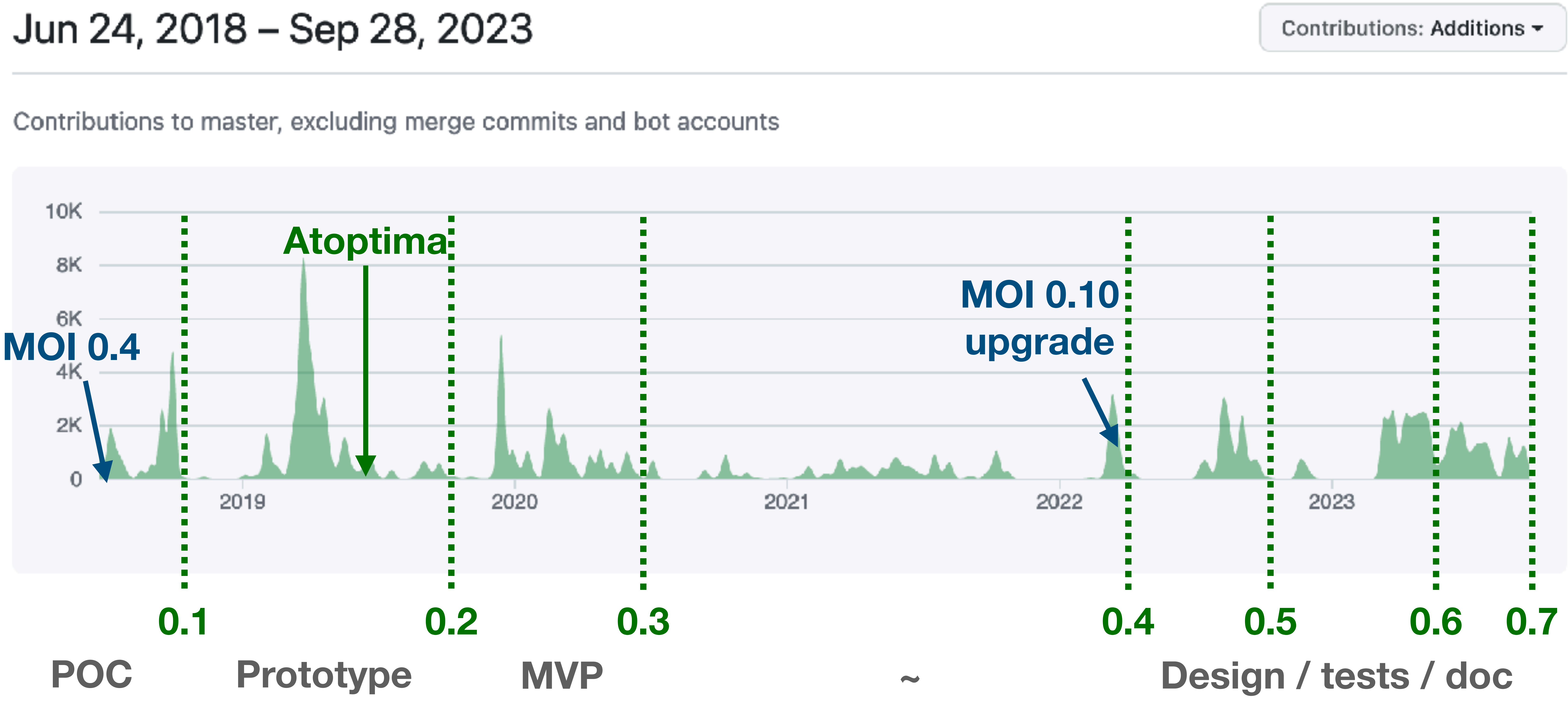
Algorithms architecture

Generic functions + Interface

Default implementation



Roadmap



- All features for the textbook Branch-Cut-and-Price are available
- Now focus on performance features

When should I give it a try?

- Julia as high-level language of your project
- Trying Dantzig-Wolfe/Benders decomposition on your problem (POC)
- Prototyping (features: callbacks, advanced parameters)
- Building a viable project (confirmed Julia devs who feel comfortable with theory)

Coluna.jl

Open-source

https://github.com/atoptima/Coluna.jl

Registered

```
] add BlockDecomposition, Coluna
```

Contributors

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References

- BONAMI, Pierre, SALVAGNIN, Domenico, et TRAMONTANI, Andrea. Implementing automatic Benders decomposition in a modern MIP solver. In : *Integer Programming and Combinatorial Optimization: 21st International Conference, IPCO 2020, London, UK, June 8–10, 2020, Proceedings*. Springer International Publishing, 2020. p. 78-90.

- ACHTERBERG, Tobias, BIXBY, Robert E., GU, Zonghao, *et al.* Presolve reductions in mixed integer programming. *INFORMS Journal on Computing*, 2020, vol. 32, no 2, p. 473-506

- Diego Pecin, Artur Pessoa, Marcus Poggi, and Eduardo Uchoa. Improved branch-cut-andprice for capacitated vehicle routing. Mathematical Programming Computation, 9(1):61–100, 2017.

- Pierre Le Bodic and George Nemhauser. An abstract model for branching and its application to mixed integer programming. Mathematical Programming, 166(1):369–405, Nov 2017.

- A. Pessoa, R. Sadykov, E. Uchoa, F. Vanderbeck. "Automation and combination of linear-programming based stabilization techniques in column generation". INFORMS Journal on Computing, 30(2):339-360, 2018.

- Marcus Poggi de Aragão and Eduardo Uchoa (2003). “Integer program reformulation for robust branch-and-cut-and-price”. In: Annals of Mathematical Programming in Rio. Ed. by Laurence A. Wolsey. Búzios, Brazil, pp. 56–61

- Mads Jepsen, Bjorn Petersen, Simon Spoorendonk, and David Pisinger (2006). A Non-Robust Branch-And-Cut-And-Price Algorithm for the Vehicle Routing Problem with Time Windows. Technical report 06/03. Dept. of Computer Science, University of Copenhagen

- Achterberg, T., 2007. Constraint integer programming. Ph.D. thesis, Technische Universitat Berlin.

- O Kullmann. Handbook of Satisfiability, chapter Fundaments of branching heuristics, pages 205–244. IOS Press, Amsterdam, 2009.

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