Polynomial Optimization

Benoît Legat

Julia and Optimization Days 2023, October 6th

ERC “Back to the Roots” with Prof. Bart De Moor, STADIUS, KU Leuven

Polynomial optimization

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & p(x) \\ \text{s.t.} \quad & h_i(x) = 0 && \forall i \in \{1,\dots,m_h\} \\ & g_i(x) \leq 0 && \forall i \in \{1,\dots,m_g\} \end{aligned}$$

where p, h_i, g_i are polynomials.

Easy or hard ?

- If p, g_i are convex and h_i are linear, it is convex...
- ... but in general, it is NP-hard
- Special case with $p = 0$ and $m_g = 0$, system of polynomial equation, already NP-hard...

Two types of Lagrangian multipliers

Constant multipliers – local certificate – KKT system

Find $x \in \mathbb{R}^n, \lambda \in \mathbb{R}^{m_h}, \sigma \in \mathbb{R}^{m_g}$ s.t.

$$\nabla \left[p(x) + \sum_{i=1}^{m_h} \lambda_i h_i(x) + \sum_{i=1}^{m_g} {\sigma_i}^2 g_i(x) \right] = 0$$

Polynomial multipliers – global certificate – Putinar

Find $\gamma \in \mathbb{R}$ and polynomials $\lambda_i(x), \sigma_{i,j}(x) \in \mathbb{R}[x]$ s.t.

$$\forall x \in \mathbb{R}^n, \quad p(x) + \sum_{i=1}^{m_h} \lambda_i(x) h_i(x) + \sum_{i=1}^{m_g} \left(\sum_j {\sigma_{i,j}}^2(x) \right) g_i(x) = \gamma$$

Solving the KKT system

Given a system $S = \{x \mid h_i(x) = 0\}$, for any $\lambda \in \mathbb{R}[x]^m$,

$$\sum_{i=1}^{m} \lambda_i(x) h_i(x) = 0, \qquad \forall x \in S.$$

This is the linear span of, for all $\alpha \in \mathbb{N}^n, i \in \{1, \dots, m\}$,

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} h_i$$

Macaulay matrix M_d has those of maxdegree below d as rows

Buchberger : Gaussian elimination → numerically unstable!

- Compute right null space Z_d with SVD → numerically stable!
- Mind the gap : rank of truncation of Z_d equal to rank of Z_d
- Gap condition is sufficient for finding the solution from Z_d.

Obtaining the minimizers for Sum-of-Squares

$$\forall x \in \mathbb{R}^n, \quad p(x) + \sum_{i=1}^{m_h} \lambda_i(x) h_i(x) + \sum_{i=1}^{m_g} \left(\sum_j {\sigma_{i,j}}^2(x) \right) g_i(x) = \gamma$$

We search over dual Lagrangian multipliers $\lambda_i(x)$, $\sigma_{i,j}(x)$. But then what's the primal x ? Isn't it the "dual of the dual" ?

- Conic dual is a symmetric PSD matrix of *moments* M_d.
- Positive semidefiniteness necessary but not sufficient for existence of a *measure* with these *moments*.
- Flatness property : Rank of truncation equal to rank of M_d
- Flatness sufficient condition for existence of an *atomic* measure with these moments.

Can you find the minimizer for the Goldstein-price function ?

```
MomentMatrix with row/column basis:
 MonomialBasis([1, x[2], x[1], x[2]^2, x[1]*x[2], x[1]^2, x[2]^3, x[1]*x[2]^2,
 x[1]^2*x[2], x[1]^3, x[2]^4, x[1]*x[2]^3, x[1]^2*x[2]^2, x[1]^3*x[2], x[1]^4])
And entries in a 15×15 SymMatrix{Float64}:
  0.999999      -0.999999       4.381387e-7  ...    1.004378e-8
 -0.999999       0.999999      -4.374988e-7         5.650864e-7
  4.381387e-7   -4.374988e-7    2.282174e-9         9.580218e-7
  0.999999      -0.999999       4.381366e-7         0.001309
 -4.374973e-7    4.381360e-7    4.364830e-10        0.001473
  2.280602e-9    4.350399e-10   3.053123e-9  ...    0.002703
 -0.999999       0.999999      -4.371625e-7         0.099747
  4.381382e-7   -4.371629e-7    1.629694e-9         0.688158
  4.349555e-10   1.628054e-9    3.198895e-9         0.497798
  3.053181e-9    3.199220e-9    1.004332e-8         1.286501
  0.999999      -0.999999       6.523214e-7  ... 1766.382361
 -4.371617e-7    6.523217e-7    2.263102e-7      -362.739066
  1.628781e-9    2.263096e-7    4.378267e-7      2468.848686
  3.198388e-9    4.378288e-7    5.650866e-7       691.956682
  1.004378e-8    5.650864e-7    9.580218e-7      4051.055429
```

Heuristic from Eq (6.15) of Laurent, M. (2009). *Sums of squares, moment matrices and optimization over polynomials*

Randomized rounding with 2nd order moments

- **0**th-order $\mathbb{E}[1]$ should be 1
- **1**th-order $\mathbb{E}[x_i]$ should be the mean μ: previous slide uses μ as candidate
- **2**th-order $\mathbb{E}[x_i x_j]$ should be the covariance Σ:

Rounded **random** sampling:

- Candidates sampled from $\mathbb{N}(\mu, \Sigma)$?
- Should also try to round it to the feasible set! E.g., for $x_i = \{-1, +1\}$, round with **sign**. Guarantees proved in [GW95].

[GW95] Goemans, M. X., Williamson, D. P. (1995). *Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming.*

How do we round for arbitrary sets ?

Example

Let P_1, P_2 be polyhedra. $x_1 \in P_1, x_2 \in P_2, a(x_2)^\top x_1 \leq \beta(x_2)$.

1. $\bar{v}_2 \leftarrow$ project v_2 to P_2
2. Project v_1 to $P_1, a(\bar{v}_2)^\top x_1 \leq \beta(\bar{v}_2)$

Algorithm

Input: Candidate v and constraints $g_i(x) \leq 0$

Partition variables x into y and z

$\bar{v}_z \leftarrow$ recursively project v_z onto $g_i(z) \leq 0$ not depending on y

$\bar{v}_y \leftarrow$ project v_y to $g_i(y, \bar{v}_z) \leq 0$ (convex thanks to partition)

return $\bar{v} \leftarrow$ combine $\bar{v}_z$ and $\bar{v}_y$

Example of rounding

$$
\begin{aligned}
\min \quad & -x-y \\
\text{s.t.} \quad & y \leq 2x^4 - 8x^3 + 8x^2 + 2 \\
& y \leq 4x^4 - 32x^3 + 88x^2 - 96x + 36 \\
& 0 \leq x \leq 3, \quad 0 \leq y \leq 4.
\end{aligned}
$$

Flatness

Round $\mu = \mathbb{E}[x_i]$

104 MONIQUE LAURENT

order t	rank sequence	bound p_t^{mom}	solution extracted
2	(1,1,4)	-7	none
3	(1,2,2,4)	-6.6667	none
4	(**1,1,1,1**,6)	-5.5080	(2.3295,3.1785)

TABLE 2
Moment relaxations for Example 6.23

bound	rounded solution
-3	(-3, 0)
-3.9012	(-2.6667, 1.2345)
-5.5080	(-2.3295, 3.1785)

Example (6.23) of Laurent, M. (2009). *Sums of squares, moment matrices and optimization over polynomials*

Polynomial optimization as nonconvex QCQP

$$\begin{aligned} \max \quad & x^6y^3 + x^3y^3 - x^3 \\ \text{s.t.} \quad & -2 \leq x \leq 3 \\ & 5 \leq y \leq 7 \\ & x^3 + y^3 \leq 3 \end{aligned}$$

y^6						
y^5						
y^4						
y^3	xy^3	x^2y^3	x^3y^3	x^4y^3	x^5y^3	x^6y^3
y^2	xy^2	x^2y^2	x^3y^2	x^4y^2	x^5y^2	x^6y^2
y	xy	x^2y	x^3y	x^4y	x^5y	x^6y
1	x	x^2	x^3	x^4	x^5	x^6

Polynomial optimization as nonconvex QCQP

$$
\begin{aligned}
\max \quad & x^6y^3 + x^3y^3 - x^3 \\
\text{s.t.} \quad & -2 \le x \le 3 \\
& 5 \le y \le 7 \\
& x^3 + y^3 \le 3
\end{aligned}
$$

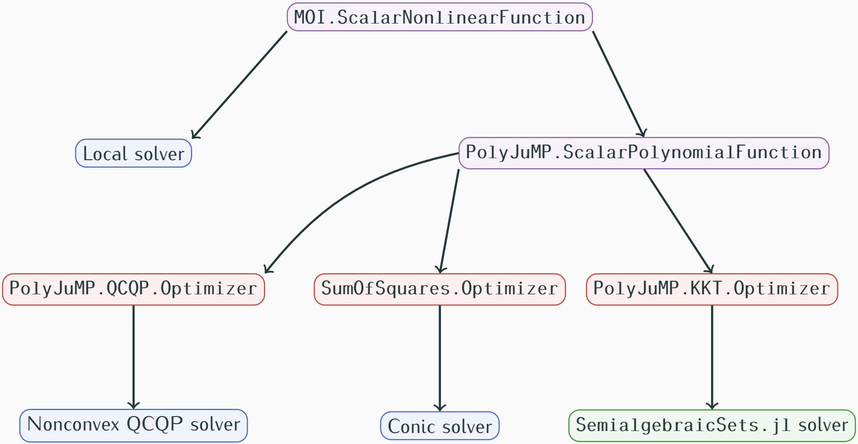
y^6						
y^5						
y^4						
y^3	xy^3	x^2y^3	x^3y^3	x^4y^3	x^5y^3	x^6y^3
y^2	xy^2	x^2y^2	x^3y^2	x^4y^2	x^5y^2	x^6y^2
y	xy	x^2y	x^3y	x^4y	x^5y	x^6y
1	x	x^2	x^3	x^4	x^5	x^6

Polynomial optimization as nonconvex QCQP

$$
\begin{aligned}
\max \quad & x^6y^3 + x^3y^3 - x^3 \\
\text{s.t.} \quad & -2 \leq x \leq 3 \\
& 5 \leq y \leq 7 \\
& x^3 + y^3 \leq 3
\end{aligned}
$$

y^6						
y^5						
y^4						
y^3	xy^3	x^2y^3	x^3y^3	x^4y^3	x^5y^3	x^6y^3
y^2	xy^2	x^2y^2	x^3y^2	x^4y^2	x^5y^2	x^6y^2
y	xy	x^2y	x^3y	x^4y	x^5y	x^6y
1	x	x^2	x^3	x^4	x^5	x^6

Polynomial Optimization Interface



Complementarity between Macaulay and Moment matrices

	Moment matrix	Macaulay matrix
Relies on	Conic solver	SVD
Fixed d	Polynomial	Polynomial
Growing d	Exponential	Exponential
M_d	Real radical	Spurious complex solutions
M_d	Complementary slackness	Spurious FOCPs
M_d	Low-accuracy system	Numerically robust

Complementarity between Macaulay and Moment matrices

	Moment matrix	Macaulay matrix
Relies on	Conic solver	SVD
Fixed d	Polynomial	Polynomial
Growing d	Exponential	Exponential
M_d	Real radical	Spurious complex solutions
M_d	Complementary slackness	Spurious FOCPs
M_d	Low-accuracy system	Numerically robust

Seems complementary, could they work together ?

Mixing Macaulay and Sum-of-Squares frameworks